Flow-forced dynamics of a triple heat exchanger—Part I: Formulation of mathematical model

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A triple heat exchanger consists of essentially three heat exchangers interconnected to form two circulating loops of fluid flow. A dynamic model in dimensionless form is formulated with the boundary conditions to suit the particular experimental triple heat exchanger described in Part II of the paper. By applying the method of weighted residual to the double-pipe heat exchangers and the upwind weighted residual method to the single tube heat exchanger and the connecting pipes, a generalized governing matrix differential equation for all types of inputs including the flow disturbances is obtained for computer simulation.

Keywords: heat exchangers; dynamic model; weighted residuals

Introduction

A triple heat exchanger essentially consists of three heat exchangers linked together by connecting pipes to form two circulating loops, one between the single tube heat exchanger (STHE) and the primary heat exchanger—named as the primary loop—and the other between primary and secondary heat exchangers—named as the carrier loop (as shown in Figure 1). In the present study the double-pipe heat exchanger (DPHE) is used for both the primary and secondary heat exchangers. The thermal energy is eventually transferred from the heat source in the STHE to the shell fluid of the secondary heat exchanger. Such heat transfer systems are widely applied in nuclear reactor power plants and solar heating systems.

In a large class of heat exchanger control problems, the fluid discharge temperature is controlled by manipulation of the flow rate of the other fluid in the system. Flow-forced dynamics of heat exchangers have therefore been intensively studied. Mozley[1] derived a five-sectioned lumped parameter model in which wall effects were ignored and heat transfer coefficient was assumed to be independent of flow-rate. He derived the transfer functions relating outlet temperature changes to tube fluid velocity change. Law[2] employed the nonlinear relationship between the heat transfer coefficient and flow-rate, and took wall effect into account in his derivation. The derivation of the transfer functions was based on the linearization of the system equations about the steady-state operating point. Later, Privott and Ferrell[3] even considered the variations of heat transfer coefficient with temperature as well as flow-rate in his nonlinear model. The nonlinear model was simplified by omitting the temperature dependence of the heat transfer coefficient and linearizing the system equations about the steady state operating point. In general, nonlinear terms due to multiplicative terms of temperatures and flow-rates exist. The linear model is rather inaccurate unless the magnitude of flow disturbance is small. In such a case, the measurements of temperature transients are difficult in view of small changes of steady-state temperatures.

Modern approaches attempt to establish system models that adapt to the use of efficient computing techniques. Early on, Chiu and Fung[4] built an experimental triple heat exchanger system to verify the theoretical results of temperature responses for sudden changes of heat input and secondary inlet temperature. They first studied the dynamics of the individual heat exchangers.[5] Because of the success of their proposed models, a similar approach was again employed. The method of weighted residuals (MWR) proposed by Finlayson[6] and applied to the heat exchanger problems of Kanoh[7] was used to formulate the dynamic models of primary and secondary heat exchangers. For the single tube heat exchanger and the four connecting pipes, their models are primarily characterized by an advective equation. In this case, the upwind weighted residual method (UWR)[8,9] is adopted.

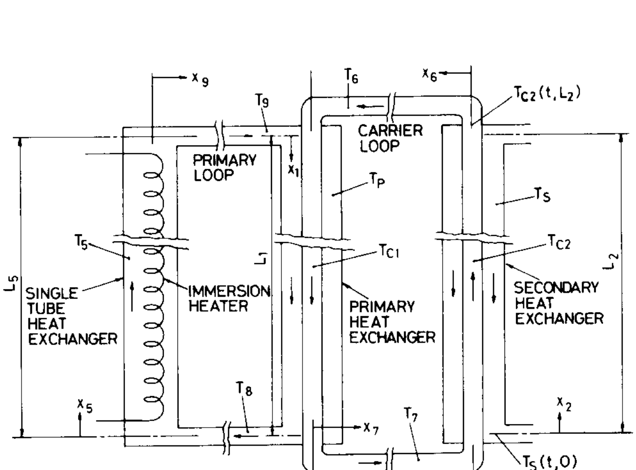

Figure 1 Schematic diagram of triple heat exchanger

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Dynamic model

The mathematical model of the triple heat exchanger as shown in Figure 1 is derived from the fundamental equations of the distributed parameter model for the individual heat exchangers and the connecting pipes. The following assumptions are made:

General—
1. the liquids are incompressible and their physical properties (specific gravity, specific heat and viscosity) are constant;
2. the physical properties (specific gravity and specific heat) of the metal walls are constant;
3. the temperature and velocity profiles of the liquids are uniform in the radial directions.

Double-pipe heat exchanger—
4. the tube wall and shell wall thermal capacitances are additive to those of the tube liquid and shell liquid respectively;
5. the thermal capacitances of the headstocks of the heat exchanger are additive to that of the shell wall;
6. the steady state overall heat transfer coefficient U between the two liquids are assumed to be of the form

$$U^{-1}=[p_1v_1^{0.8}]^{-1}+[p_2v_2^{0.8}]^{-1}$$

where p_1 and p_2 are constants;
7. the change of overall heat transfer coefficient for a change of flow-rate is proportional to the change of flow-rate;
8. the heat loss rate to the surroundings is proportional to the temperature difference between the liquid and the surroundings. This is experimentally verified to be correct.

Single tube heat exchanger—
9. the thermal capacitance of the metal wall is additive to that of the liquid;
10. the heat loss rate to the surroundings is proportional to the power of temperature difference between the liquid and the surroundings. This is experimentally verified to be correct for the particular installation of the immersion heaters from a number of steady state temperature readings taken under different power input and flow conditions;
11. the heater dynamics are negligible.

Notation

A, B	31 × 31 matrices
a_i	Dimensionless coefficients, $i=1$ to 6
b_i	Dimensionless coefficients, $i=1, 2$
C_i	Intermediate constants, $i=1$ to 5
c, c_c, c_5, c_w	Specific heat capacities of liquid and metal in connecting pipes, and liquid and metal wall in STHE
c_1, c_2, c_s, c_t	Specific heat capacities of shell and tube fluids, shell and tube walls
D_i	3 × 1 matrices, $i=3$ to 7
F_1, F_2	Functions defined by Equations 17 and 18
f, f_1, f_2, f_5	Dimensionless parameters
h_L	Heat loss coefficient in a heat exchanger
k_1, k_2	Constant coefficients
ι	Dimensionless distance variable $=2(x_i/L_i)-1$
L_i	Lengths of various components in the triple heat exchanger corresponding to x_i, $i=1, 2, 5, \ldots, 9$
M, M_1, M_2, M_5	Masses per unit length of liquid in connecting pipe, shell and tube liquids (DPHE), and liquid in STHE respectively
M_c, M_s, M_t, M_w	Equivalent masses per unit length of connecting pipe wall, shell and tube walls (DPHE), and metal wall of STHE respectively
$m_{i,j}$	Normalized heat transfer constants $i, j=1, 2$
$\dot{m}_1, \dot{m}_2, \dot{m}_5$	Mass flow rates of shell and tube liquids (DPHE), and liquid of STHE respectively
N_i	31 × 31 matrices, $i=1, 2, 3$
P_i	31 × 1 matrices, $i=1, \ldots, 5$
q	Heating rate per unit length
$\bar{r}_1, \bar{r}_2, \bar{r}_3$	Dimensionless ratios of velocities in the triple heat exchanger
r_4	Outer radius of the outer tube in DPHE
r_m	Mean radius of the inner tube in DPHE
S_i	Matrices, $i=1, 2, 3$
T_i	Matrices, $i=1, 2, 3, 4$
$T_i, T_p, T_{c1}, T_{c2}, T_s$	Liquid temperatures at various locations in the triple heat exchanger, $i=5, \ldots, 9$
T_a	Ambient temperature
T_{is}	Initial steady state temperature, $i=$ index − variable
$T_{\sin}$	Inlet temperature of secondary heat exchanger
$\bar{T}$	Liquid temperature as function of τ and ι
t	Real time
U	4 × 1 column vector with elements in Legendre polynomials
U_1, U_2	Overall heat transfer coefficients of the primary and secondary heat exchangers
u_i	Dimensionless disturbances, $i=1, \ldots, 11$
$\mathsf{V}_1, \mathsf{V}_2$	Matrices defined in Equations 21 and 22
v_1, v_2, v_5	Velocities of shell and tube liquids (DPHE) and liquid of STHE respectively
v_i	Velocities in various pipes, $i=6, \ldots, 9$
X_i	Temperature vectors, $i=1, \ldots, 9$
X	31 × 1 temperature vector in Equation 33
x_i	Distance variables in the triple heat exchanger $i=1, 2, 5, \ldots, 9$
α	Dimensionless parameter
δ	Delta function
θ_i	Dimensionless temperature deviations of liquids, $i=1, \ldots, 9$
τ	Dimensionless time $=t/[L_1/v_{2,1}(0)]$
τ_i	Dimensionless pure time delays, $i=5, \ldots, 9$

Subscripts

{ }	Vector
[ ]	Matrix
,1	Primary heat exchanger
,2	Secondary heat exchanger
i, j, k	Index variables

Connecting pipes—

12. the thermal capacitance of the metal wall is additive to that of liquid;
13. the heat loss is negligible.

Under these assumptions, the resulting mathematical model for the triple heat exchanger is established from component equations with the boundary conditions chosen to suit the particular physical requirements. For the equations derived below, the second subscript in double subscripts separated by a comma indicates the primary (1) or secondary (2) heat exchanger.

$$\frac{\partial T_p}{\partial x_1}+\frac{(1+f_{1,1})}{v_{1,1}}\frac{\partial T_p}{\partial t}=\frac{2\pi r_{m,1}U_1}{\dot m_{1,1}c_{1,1}}(T_{c1}-T_p)+\frac{2\pi r_{4,1}h_{L,1}}{\dot m_{1,1}c_{1,1}}(T_a-T_p) \tag{1}$$

$$\frac{\partial T_{c1}}{\partial x_1}+\frac{(1+f_{2,1})}{v_{2,1}}\frac{\partial T_{c1}}{\partial t}=\frac{2\pi r_{m,1}U_1}{\dot m_{2,1}c_{2,1}}(T_p-T_{c1}) \tag{2}$$

$$-\frac{\partial T_s}{\partial x_2}+\frac{(1+f_{1,2})}{v_{1,2}}\frac{\partial T_s}{\partial t}=\frac{2\pi r_{m,2}U_2}{\dot m_{1,2}c_{1,2}}(T_{c2}-T_s)+\frac{2\pi r_{4,2}h_{L,2}}{\dot m_{1,2}c_{1,2}}(T_a-T_s) \tag{3}$$

$$\frac{\partial T_{c2}}{\partial x_2}+\frac{(1+f_{2,2})}{v_{2,2}}\frac{\partial T_{c2}}{\partial t}=\frac{2\pi r_{m,2}U_2}{\dot m_{2,2}c_{2,2}}(T_s-T_{c2}) \tag{4}$$

$$\frac{\partial T_5}{\partial x_5}+\frac{(1+f_5)}{v_5}\frac{\partial T_5}{\partial t}=\frac{q}{\dot m_5c_5}-\frac{k_2(T_5-T_a)^{k_1}}{\dot m_5c_5} \tag{5}$$

$$\frac{\partial T_i}{\partial x_i}+\frac{(1+f)}{v_i}\frac{\partial T_i}{\partial t}=0,\qquad i=6,7,8,9 \tag{6}$$

The boundary conditions are

$$T_i(t,0)=T_j(t,L_k) \tag{7a}$$

for $i=5, 6, 7, 8, 9, p, c1, c2$, and corresponding $j=8, c2, c1, p, 5, 9, 6, 7$, and $k=8, 2, 1, 1, 5, 9, 6, 7$

$$T_s(t,L_2)=T_{\sin}(t) \tag{7b}$$

The initial conditions are

$$T_i(0,x_j)=T_{is}(x_j)\qquad \text{for appropriate } i, j \tag{8}$$

Problem formulation and solution

Although in the present study only three flow disturbance inputs—namely primary loop water flow-rate, carrier loop water flow-rate and secondary inlet flow-rate—are studied, it will be useful for the purpose of generality to include the other two inputs, namely secondary inlet temperature and power input, in the analysis. To put this model in a general dimensionless form, the following dimensionless variables are defined.

$$a_1=\frac{2\pi r_{m,1}U_1[v_{1,1}(0),v_{2,1}(0)]L_1}{\dot m_{1,1}(0)c_{1,1}}$$

$$a_2=\frac{2\pi r_{m,1}U_1[v_{1,1}(0),v_{2,1}(0)]L_1}{\dot m_{2,1}(0)c_{2,1}}$$

$$a_3=\frac{2\pi r_{m,2}U_2[v_{1,2}(0),v_{2,2}(0)]L_2}{\dot m_{1,2}(0)c_{1,2}}$$

$$a_4=\frac{2\pi r_{m,2}U_2[v_{1,2}(0),v_{2,2}(0)]L_2}{\dot m_{2,2}(0)c_{2,2}}$$

$$a_5=\frac{2\pi r_{4,1}h_{L,1}L_1}{\dot m_{1,1}(0)c_{1,1}}\qquad a_6=\frac{2\pi r_{4,2}h_{L,2}L_2}{\dot m_{1,2}(0)c_{1,2}}$$

$$\bar r_1=\frac{v_{2,1}(0)}{v_{1,1}(0)}\qquad \bar r_2=\frac{v_{2,1}(0)L_2}{v_{1,2}(0)L_1}$$

$$\bar r_3=\frac{v_{2,1}(0)L_2}{v_{2,2}(0)L_1}\qquad f=\frac{M_cc_c}{Mc}$$

$$f_{1,i}=\frac{M_{s,i}c_{s,i}}{M_{1,i}c_{1,i}},\quad i=1,2\qquad f_{2,i}=\frac{M_{t,i}c_{t,i}}{M_{2,i}c_{2,i}},\quad i=1,2$$

$$f_5=\frac{M_wc_w}{M_5c_5}$$

$$\theta_1=\frac{\bar T_p(\tau,\iota)-\bar T_p(0,\iota)}{\bar T_p(0,-1)-\bar T_{c1}(0,-1)}\qquad \theta_2=\frac{\bar T_{c1}(\tau,\iota)-\bar T_{c1}(0,\iota)}{\bar T_p(0,-1)-\bar T_{c1}(0,-1)}$$

$$\theta_3=\frac{\bar T_s(\tau,\iota)-\bar T_s(0,\iota)}{\bar T_{c2}(0,-1)-\bar T_s(0,1)}\qquad \theta_4=\frac{\bar T_{c2}(\tau,\iota)-\bar T_{c2}(0,\iota)}{\bar T_{c2}(0,-1)-\bar T_s(0,1)}$$

$$\theta_i=\frac{\bar T_i(\tau,\iota)-\bar T_i(0,\iota)}{\bar T_p(0,-1)-\bar T_{c1}(0,-1)},\qquad i=5,7,8,9$$

$$\theta_6=\frac{\bar T_6(\tau,\iota)-\bar T_6(0,\iota)}{\bar T_{c2}(0,-1)-\bar T_s(0,1)}$$

$$u_1=\frac{v_{1,1}(\tau)-v_{1,1}(0)}{v_{1,1}(0)}\qquad u_2=\frac{v_{2,1}(\tau)-v_{2,1}(0)}{v_{2,1}(0)}$$

$$u_3=\frac{v_{1,2}(\tau)-v_{1,2}(0)}{v_{1,2}(0)}\qquad u_4=\frac{v_{2,2}(\tau)-v_{2,2}(0)}{v_{2,2}(0)}$$

$$u_i=\frac{v_i(\tau)-v_i(0)}{v_i(0)},\qquad i=5 \text{ to } 9$$

$$u_{10}=\frac{q(\tau)-q(0)}{q(0)}\qquad u_{11}=\frac{\bar T_s(\tau,1)-\bar T_s(0,1)}{\bar T_{c2}(0,-1)-\bar T_s(0,1)}$$

$$m_{i,1}=\frac{U_1[v_{i,1}(\infty),v_{k,1}(0)]-U_1[v_{1,1}(0),v_{2,1}(0)]}{u_i(\infty)U_1[v_{1,1}(0),v_{2,1}(0)]},\qquad i=1,2$$

with corresponding $k=2, 1$; and $m_{i,1}=0$ if $u_i=0$

$$m_{j,2}=\frac{U_2[v_{j,1}(\infty),v_{k,2}(0)]-U_2[v_{1,2}(0),v_{2,2}(0)]}{u_{j+2}(\infty)U_2[v_{1,2}(0),v_{2,2}(0)]},\qquad j=1,2$$

with corresponding $k=2, 1$; and $m_{j,2}=0$ if $u_{j+2}=0$

$$\tau_i=\frac{L_i/v_i(0)}{L_1/v_{2,1}(0)},\qquad i=5 \text{ to } 9\qquad \alpha=\frac{\bar T_p(0,-1)-\bar T_{c1}(0,-1)}{\bar T_{c2}(0,-1)-\bar T_s(0,1)}$$

In deriving the initial steady-state temperature profiles additional assumptions are made that, for each DPHE, the heat loss term is negligible; for the STHE, $T_5(0,x_5)$ is a linear function of x_5. The governing partial differential equations are as follows

$$\begin{aligned}2(1+u_1)\frac{\partial\theta_1}{\partial\iota}+\bar r_1(1+f_{1,1})\frac{\partial\theta_1}{\partial\tau}&=a_1(1+m_{1,1}u_1+m_{2,1}u_2)(\theta_2-\theta_1)\\&\quad-a_5\theta_1-a_1(m_{1,1}-1)\,e^{-(a_2+a_1)(\iota+1)/2}u_1\\&\quad-a_1m_{2,1}\,e^{-(a_2+a_1)(\iota+1)/2}u_2\\&\quad+2(1+u_1)\,\delta(\iota+1)\theta_9(\tau,1)\end{aligned} \tag{9}$$

$$2(1+u_2)\frac{\partial\theta_2}{\partial\iota}+(1+f_{2,1})\frac{\partial\theta_2}{\partial\tau}$$
$$=a_2(1+m_{1,1}u_1+m_{2,1}u_2)(\theta_1-\theta_2)$$
$$+a_2(m_{2,1}-1)\,e^{-(a_2+a_1)(\iota+1)/2}u_2$$
$$+a_2m_{1,1}\,e^{-(a_2+a_1)(\iota+1)/2}u_1$$
$$+2(1+u_2)\,\delta(\iota+1)\theta_6(\tau,1)/\alpha \quad (10)$$

$$-2(1+u_3)\frac{\partial\theta_3}{\partial\iota}+\bar{r}_2(1+f_{1,2})\frac{\partial\theta_3}{\partial\tau}$$
$$=a_3(1+m_{1,2}u_3+m_{2,2}u_4)(\theta_4-\theta_3)$$
$$-a_6\theta_3+\frac{a_3(a_4-a_3)(m_{1,2}-1)\,e^{-(a_4-a_3)(\iota+1)/2}}{a_4-a_3\,e^{-(a_4-a_3)}}u_3$$
$$+\frac{a_3m_{2,2}(a_4-a_3)\,e^{-(a_4-a_3)(\iota+1)/2}}{a_4-a_3\,e^{-(a_4-a_3)}}u_4$$
$$+2(1+u_3)\,\delta(\iota-1)u_{11} \quad (11)$$

$$2(1+u_4)\frac{\partial\theta_4}{\partial\iota}+\bar{r}_3(1+f_{2,2})\frac{\partial\theta_4}{\partial\tau}$$
$$=a_4(1+m_{1,2}u_3+m_{2,2}u_4)(\theta_3-\theta_4)$$
$$-\frac{a_4(a_4-a_3)m_{1,2}\,e^{-(a_4-a_3)(\iota+1)/2}}{a_4-a_3\,e^{-(a_4-a_3)}}u_3$$
$$-\frac{(a_4-a_3)(m_{2,2}-1)a_4\,e^{-(a_4-a_3)(\iota+1)/2}}{a_4-a_3\,e^{-(a_4-a_3)}}u_4$$
$$+2(1+u_4)\,\delta(\iota+1)\theta_7(\tau,1)\alpha \quad (12)$$

$$2(1+u_5)\frac{\partial\theta_5}{\partial\iota}+\tau_5(1+f_5)\frac{\partial\theta_5}{\partial\tau}+F_1(\iota)\theta_5+F_2(\iota)u_5=C_3u_{10} \quad (13)$$

$$2(1+u_i)\frac{\partial\theta_i}{\partial\iota}+\tau_i(1+f)\frac{\partial\theta_i}{c\tau}=0 \qquad \text{for } i=6,7,8,9 \quad (14)$$

Boundary conditions:

$$\theta_i(\tau,-1)=0, \qquad i=1,2,4 \quad (15a)$$
$$\theta_i(\tau,-1)=\theta_j(\tau,1)$$
$$\text{for } i=5,6,7,8,9 \quad (15b)$$

and corresponding $j=8, 4, 2, 1, 5$.

$$\theta_3(\tau,1)=0 \quad (15c)$$

Initial conditions:

$$\theta_i(0,\iota)=0 \qquad \text{for } i=1 \text{ to } 9 \quad (16)$$

where

$$F_1(\iota)=\left[C_1+C_2\left(\frac{\iota+1}{2}\right)\right]^{k_1-1} \quad (17)$$

$$F_2(\iota)=C_3-\left[C_4+C_5\left(\frac{\iota+1}{2}\right)\right]^{k_1} \quad (18)$$

and C_1 to C_5 are intermediate constants in terms of temperatures and physical parameters of the STHE and the liquid, shown in Ref. 4 or in Appendix 1.

The MWR is applied to Equations 9–12 with Legendre polynomials chosen as the weighting functions for their orthogonality properties. The procedure for deriving partial differential equations and formulating the matrix differential equations for the parallel-flow heat exchanger is shown in Appendix 2. The following approximate solutions are assumed

$$\theta_i(\tau,\iota)=(\iota+1)\mathrm{U}'(\iota)\mathrm{X}_i(\tau), \qquad i=1,2,4 \quad (19a)$$
$$\theta_3(\tau,\iota)=(\iota-1)\mathrm{U}'(\iota)\mathrm{X}_3(\tau) \quad (19b)$$

which satisfy the boundary conditions specified by Equations 15a and 15c, and

$$\mathrm{U}'(\iota)=[1,\iota,(3\iota^2-1)/2,(5\iota^3-3\iota)/2] \quad (20)$$

For Equations 13 and 14 the UWR is applied. The procedures for formulating the matrix differential equations for a single tube heat exchanger and a connecting pipe are shown in Appendix 1. Defining

$$\mathrm{V}_1=\int_{-1}^{1}\mathrm{U}(\iota)\,e^{-(a_2+a_1)(\iota+1)/2}\,d\iota \quad (21)$$

$$\mathrm{V}_2=\int_{-1}^{1}\mathrm{U}(\iota)\,e^{-(a_4-a_3)(\iota+1)/2}\,d\iota \quad (22)$$

$$b_1=\frac{a_3(a_4-a_3)}{a_4-a_3\,e^{-(a_4-a_3)}} \quad (23)$$

$$b_2=\frac{a_4(a_4-a_3)}{a_4-a_3\,e^{-(a_4-a_3)}} \quad (24)$$

we obtain the following matrix equations

$$[\bar{r}_1(1+f_{1,1})\mathrm{S}_1]\{\dot{\mathrm{X}}_1\}+[2\mathrm{T}_1+(a_1+a_5)\mathrm{S}_1]\{\mathrm{X}_1\}+[-a_1\mathrm{S}_1]\{\mathrm{X}_2\}$$
$$+[2\mathrm{T}_1+a_1m_{1,1}\mathrm{S}_1]\{\mathrm{X}_1\}u_1+[-a_1m_{1,1}\mathrm{S}_1]\{\mathrm{X}_2\}u_1$$
$$+[a_1m_{2,1}\mathrm{S}_1]\{\mathrm{X}_1\}u_2+[-a_1m_{2,1}\mathrm{S}_1]\{\mathrm{X}_2\}u_2$$
$$+[a_1(m_{1,1}-1)\mathrm{V}_1]u_1+[a_1m_{2,1}\mathrm{V}_1]u_2$$
$$=2(1+u_1)\mathrm{U}(-1)\theta_9(\tau,1) \quad (25)$$

$$[(1+f_{2,1})\mathrm{S}_1]\{\dot{\mathrm{X}}_2\}+[2\mathrm{T}_1+a_2\mathrm{S}_1]\{\mathrm{X}_2\}+[-a_2\mathrm{S}_1]\{\mathrm{X}_1\}$$
$$+[-a_2m_{1,1}\mathrm{S}_1]\{\mathrm{X}_1\}u_1+[a_2m_{1,1}\mathrm{S}_1]\{\mathrm{X}_2\}u_1$$
$$+[-a_2m_{2,1}\mathrm{S}_1]\{\mathrm{X}_1\}u_2+[2\mathrm{T}_1+a_2m_{2,1}\mathrm{S}_1]\{\mathrm{X}_2\}u_2$$
$$+[-a_2m_{1,1}\mathrm{V}_1]u_1+[-a_2(m_{2,1}-1)\mathrm{V}_1]u_2$$
$$=2(1+u_2)\mathrm{U}(-1)\theta_6(\tau,1)/\alpha \quad (26)$$

$$[\bar{r}_2(1+f_{1,2})\mathrm{S}_2]\{\dot{\mathrm{X}}_3\}+[-2\mathrm{T}_2+(a_3+a_6)\mathrm{S}_2]\{\mathrm{X}_3\}$$
$$+[-a_3\mathrm{S}_1]\{\mathrm{X}_4\}+[-2\mathrm{T}_2+a_3m_{1,2}\mathrm{S}_2]\{\mathrm{X}_3\}u_3$$
$$+[-a_3m_{1,2}\mathrm{S}_1]\{\mathrm{X}_4\}u_3+[a_3m_{2,2}\mathrm{S}_2]\{\mathrm{X}_3\}u_4$$
$$+[-a_3m_{2,2}\mathrm{S}_1]\{\mathrm{X}_4\}u_4$$
$$+[b_1(m_{1,2}-1)\mathrm{V}_2]u_3+[b_1m_{2,2}\mathrm{V}_2]u_4$$
$$=2(1+u_3)\mathrm{U}(1)u_{11} \quad (27)$$

$$[\bar{r}_3(1+f_{2,2})\mathrm{S}_1]\{\dot{\mathrm{X}}_4\}+[2\mathrm{T}_1+a_4\mathrm{S}_1]\{\mathrm{X}_4\}+[-a_4\mathrm{S}_2]\{\mathrm{X}_3\}$$
$$+[-a_4m_{1,2}\mathrm{S}_2]\{\mathrm{X}_3\}u_3+[a_4m_{1,2}\mathrm{S}_1]\{\mathrm{X}_4\}u_3$$
$$+[-a_4m_{2,2}\mathrm{S}_2]\{\mathrm{X}_3\}u_4+[2\mathrm{T}_1+a_4m_{2,2}\mathrm{S}_1]\{\mathrm{X}_4\}u_4$$
$$+[-b_2m_{1,2}\mathrm{V}_2]u_3+[-b_2(m_{2,2}-1)\mathrm{V}_2]u_4$$
$$=2(1+u_4)\mathrm{U}(-1)\theta_7(\tau,1)\alpha \quad (28)$$

$$\left[\frac{(1+f_5)\tau_5}{9}\mathrm{S}_3\right]\{\dot{\mathrm{X}}_5\}+[\mathrm{T}_4]\{\mathrm{X}_5\}+[\mathrm{T}_3]\{\mathrm{X}_5\}u_5+[\mathrm{D}_7]u_5$$
$$+[\mathrm{D}_5]\theta_8(\tau,1)+[\mathrm{D}_3]u_5\theta_8(\tau,1)+\left[\frac{(1+f_5)\tau_5}{9}\mathrm{D}_4\right]\dot{\theta}_8(\tau,1)$$
$$=[\mathrm{D}_6]u_{10} \quad (29)$$

$$\left[(1+f)\frac{\tau_i}{9}S_3\right]\{\dot{X}_i\}+[T_3]\{X_i\}+[T_3]\{X_i\}u_i+[D_3]\theta_j(\tau,1)$$
$$+[D_3]u_i\theta_j(\tau,1)+\left[(1+f)\frac{\tau_i}{9}D_4\right]\dot{\theta}_j(\tau,1)=0 \quad (30)$$

for $i=6, 7, 8, 9$ and corresponding $j=4, 2, 1, 5$; where

$$X_i=[x_{1i}, x_{2i}, x_{3i}, x_{4i}]', \qquad i=1 \text{ to } 4, \quad (31)$$

and

$$X_i=[\theta_i(\tau,-\tfrac{1}{3}), \theta_i(\tau,\tfrac{1}{3}), \theta_i(\tau,1)]', \qquad i=5 \text{ to } 9 \quad (32)$$

The matrices S's, T's and D_3 to D_6 are defined in Ref. 4.

For the primary circulation loop, $u_1=u_5=u_8=u_9$, and for the carrier circulation loop, $u_2=u_4=u_5=u_7$.

By combining Equations 25–30, the resulting matrix differential equation governing a triple heat exchanger is obtained as follows

$$[A]\{\dot{X}\}+[B]\{X\}+[N_1]\{X\}u_1+[N_2]\{X\}u_2+[N_3]\{X\}u_3$$
$$+[P_1]u_1+[P_2]u_2+[P_3]u_3+[P_4]u_{10}+[P_5]u_{11}=0 \quad (33)$$

It should be noted that the derivation of constants $m_{1,1}$ to $m_{2,2}$ requires the plant to be subjected to only one of the liquid flow rate disturbances at a time. However other combinations of disturbances may occur concurrently.

Conclusions

A general mathematical model of the triple heat exchanger is developed by including the heat loss and assuming that the wall capacitance is additive to that of the fluid thermal capacitance in each of the heat exchangers and connecting pipes. In comparison with a model which includes the wall dynamics, the proposed one reduces the model order by 50% and thus shortens the computing time by 75%. However this procedure is only accurate provided that the wall capacitance is small in comparison with its respective fluid capacitance. Further reduction of computing time is achieved by assuming that the change of the overall heat transfer coefficient for a change of the fluid flow-rate is proportional to the change of the flow rate. In this case the matrices in the governing matrix equation are constant and their values require no repeated updating. Since the heat exchangers are properly insulated, the heat loss terms are usually small. Therefore the main purpose of inclusion of the heat loss terms in the triple heat exchanger is to obtain accurate steady-state temperatures. Their effects on the dynamic responses are expected to be minimal. Numerical solutions and experimental validation of the model are presented in Part II of this paper.

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The details of the matrices of Equation 33 will be given by the authors upon request.

Appendix 1

Solution of equation for single tube heat exchanger

The governing partial differential equation for the single tube heat exchanger is shown in Equation 13. The boundary condition is

$$\theta_5(\tau,-1)=\theta_8(\tau,1)$$

The initial condition is

$$\theta_5(0,\iota)=0$$

Using the UWR, an approximate solution of the following form is assumed

$$\theta(\tau,\iota)=\phi_{i-1}(\iota)\theta(\tau,\iota_{i-1})+\phi_i(\iota)\theta(\tau,\iota_i) \quad \text{for } \iota\in(\iota_{i-1},\iota_i)$$

where

$$\phi_{i-1}=1-(\iota-\iota_{i-1})/\bar{h}$$
$$\phi_i=(\iota-\iota_{i-1})/\bar{h}$$
$$\bar{h}=(\iota_i-\iota_{i-1})$$

In the UWR formulation, the following weighting functions are used

$$w_{i-1}=\phi_{i-1}+p\beta(\iota)$$
$$w_i=\phi_i-p\beta(\iota)$$

where

$$\beta(\iota)=3(\iota-\iota_{i-1})(\iota-\iota_{i-1}-\bar{h})/\bar{h}^2 \qquad \iota\in(\iota_{i-1},\iota_i)$$

p is the parameter which lies between 0 and 1. For the present work, $p=0.4$.

The residual $R_{5,i}$ for the segment of STHE between ι_{i-1} and ι_i, where

$$R_{5,i}(\theta_5)=2(1+u_5)\frac{d\phi_{i-1}}{d\iota}\theta_5(\tau,\iota_{i-1})+2(1+u_5)\frac{d\phi_i}{d\iota}\phi_5(\tau,\iota_i)$$
$$+\tau_5(1+f_5)\phi_{i-1}\dot{\theta}_5(\tau,\iota_{i-1})+\tau_5(1+f_5)\phi_i\dot{\theta}_5(\tau,\iota_i)$$
$$+F_1(\iota)\phi_{i-1}\theta_5(\tau,\iota_{i-1})+F_1(\iota)\phi_i\theta_5(\tau,\iota_i)$$
$$+F_2(\iota)u_5-C_3u_{10} \quad (34)$$

is orthogonalized with respect to the weighting functions w_{i-1} and w_i to yield

$$\int_{\iota_{i-1}}^{\iota_i} w_{i-1}R_{5,i}\,d\iota=0 \quad (35)$$

and

$$\int_{\iota_{i-1}}^{\iota_i} w_iR_{5,i}\,d\iota=0 \quad (36)$$

Substituting Equation 34 to Equations 35 and 36 and evaluating the integrals, Equations 37 and 38 are obtained.

$$\therefore\ (1+u_5)(-1+p)\theta_5(\tau, \iota_{i-1})+(1+u_5)(1-p)\theta_5(\tau, \iota_i)$$
$$+\frac{\tau_5}{9}(1+f_5)\left(2-\frac{3p}{2}\right)\dot{\theta}_5(\tau, \iota_{i-1})+\frac{\tau_5}{9}(1+f_5)\left(1-\frac{3p}{2}\right)\dot{\theta}_5(\tau, \iota_i)$$
$$+c_{i,11}\theta_5(\tau, \iota_{i-1})+c_{i,12}\theta_5(\tau, \iota_i)+e_{i,11}u_5$$
$$-C_3\frac{2}{3}\left(\frac{1}{2}-\frac{p}{2}\right)u_{10}=0 \qquad (37)$$

$$\therefore\ (1+u_5)(-1-p)\theta_5(\tau, \iota_{i-1})+(1+u_5)(1+p)\theta_5(\tau, \iota_i)$$
$$+\frac{\tau_5}{9}(1+f_5)\left(1+\frac{3p}{2}\right)\dot{\theta}_5(\tau, \iota_{i-1})+\frac{\tau_5}{9}(1+f_5)\left(2+\frac{3p}{2}\right)\dot{\theta}_5(\tau, \iota_i)$$
$$+c_{i,21}\theta_5(\tau, \iota_{i-1})+c_{i,22}\theta_5(\tau, \iota_i)+e_{i,21}u_5$$
$$-C_3\frac{2}{3}\left(\frac{1}{2}+\frac{p}{2}\right)u_{10}=0 \qquad (38)$$

If the STHE is partitioned into three segments of equal length, a total of six such equations is obtained. By discarding the first equation, adding the second and third and the fourth and fifth, and utilizing the sixth, Equation 29 is obtained, where

$$C_4=\left\{\frac{k_2L_5}{\dot{m}_5c_5[T_p(0,0)-T_{c1}(0,0)]}\right\}^{1/k_1}[T_5(0,0)-T_a]$$

$$C_5=\left\{\frac{k_2L_5}{\dot{m}_5c_5[T_p(0,0)-T_{c1}(0,0)]}\right\}^{1/k_1}[T_5(0,L_5)-T_5(0,0)]$$

$$\mathrm{E}_i=\begin{bmatrix}e_{i,11}\\ e_{i,21}\end{bmatrix}=\int_{\iota_{i-1}}^{\iota_i}\begin{bmatrix}w_{i-1}\\ w_i\end{bmatrix}F_2(\iota)\,d\iota, \qquad i=1, 2, 3$$

$$D_7=[(e_{1,21}+e_{2,11}), (e_{2,21}+e_{3,11}), e_{3,21}]'$$

Other relevant coefficients, matrices and their elements are given in Ref. 4.

The solution of Equation 14 describing the pipe dynamics is a simplified version of the above derivation. By eliminating the terms involving F_1, F_2, and C_3, and replacing f_5 by f, Equation 30 is obtained.

Appendix 2

Formulation and solution of equations for parallel-flow heat exchanger

Steady state temperature profiles are first derived by neglecting the heat loss effect. From Equations 1 and 2, put $\partial T/\partial t=0$ and $h_{L,1}=0$

$$\frac{dT_p}{dx_1}=\frac{a_1}{L_1}(T_{c1}-T_p) \qquad (39)$$

$$\frac{dT_{c1}}{dx_1}=\frac{a_2}{L_1}(T_p-T_{c1}) \qquad (40)$$

Using the Laplace Transform method, the solutions are obtained as follows:

$$T_{c1}(0,x_1)=\frac{a_2[1-e^{-(a_1+a_2)x_1/L_1}]}{a_1+a_2}T_p(0,0)$$
$$+\left[1+\frac{a_2}{a_1}e^{-(a_1+a_2)x_1/L_1}\right]\frac{a_1}{a_1+a_2}T_{c1}(0,0) \qquad (41)$$

$$T_p(0,x_1)=\frac{a_1}{a_1+a_2}[1-e^{-(a_1+a_2)x_1/L_1}]T_{c1}(0,0)$$
$$+\left[1+\frac{a_1}{a_2}e^{-(a_1+a_2)x_1/L_1}\right]\frac{a_2}{a_1+a_2}T_p(0,0) \qquad (42)$$

Using small perturbation technique and changing $v_{1,1}$ etc. to $v_{1,1}+\Delta v_{1,1}$ etc., and U_1 to

$$U_1+\Delta U_1'+\Delta U_1''=U_1(1+m_{1,1}\,\Delta u_1+m_{2,1}\,\Delta u_2)$$

the following equation can be obtained from Equation 1

$$(1+u_1)\frac{\partial \Delta T_p}{\partial x_1}+u_1\frac{\partial T_p}{\partial x_1}+\left(\frac{1+f_{1,1}}{v_{1,1}}\right)\frac{\partial \Delta T_p}{\partial t}$$
$$=\frac{2\pi r_{m,1}U_1(1+m_{1,1}u_1+m_{2,1}u_2)}{\dot{m}_{1,1}c_{1,1}}(\Delta T_{c1}-\Delta T_p)$$
$$+\frac{2\pi r_{m,1}U_1m_{1,1}u_1}{\dot{m}_{1,1}c_{1,1}}(T_{c1}-T_p)+\frac{2\pi r_{m,1}U_1m_{2,1}u_2}{\dot{m}_{1,1}c_{1,1}}(T_{c1}-T_p)$$
$$+\frac{2\pi r_{4,1}h_{L,1}}{\dot{m}_{1,1}c_{1,1}}(-\Delta T_p)+(1+u_1)\,\delta(x_1)\,\Delta T_9(t, L_9) \qquad (43)$$

with boundary condition $\Delta T_p(t, 0)=0$.

Defining

$$\theta_1=\frac{\Delta T_p}{T_p(0,0)-T_{c1}(0,0)}$$

$$\theta_2=\frac{\Delta T_{c1}}{T_p(0,0)-T_{c1}(0,0)}$$

$$\theta_9(\tau, 1)=\frac{\Delta T_9(t, L_9)}{T_p(0,0)-T_{c1}(0,0)}$$

Equation 43 can be reduced to

$$2(1+u_1)\frac{\partial\theta_1}{\partial\iota}-u_1a_1\,e^{-(a_1+a_2)(\iota+1)/2}+\bar{r}_1(1+f_{1,1})\frac{\partial\theta_1}{\partial\tau}$$
$$=a_1(1+m_{1,1}u_1+m_{2,1}u_2)(\theta_2-\theta_1)-a_1m_{1,1}u_1\,e^{-(a_1+a_2)(\iota+1)/2}$$
$$-a_1m_{2,1}u_2\,e^{-(a_1+a_2)(\iota+1)/2}-a_5\theta_1+2(1+u_1)\,\delta(\iota+1)\theta_9(\tau, \iota) \qquad (44)$$

with boundary condition $\theta_1(\tau, -1)=0$.

Rearranging Equation 44, Equation 9 is obtained. Equation 10 can be derived by a similar method. The following equation can be obtained from Equation 2

$$(1+u_2)\frac{\partial \Delta T_{c1}}{\partial x_1}+u_2\frac{\partial T_{c1}}{\partial x_1}+\left(\frac{1+f_{2,1}}{v_{2,1}}\right)\frac{\partial \Delta T_{c1}}{\partial t}$$
$$=\frac{2\pi r_{m,1}U_1(1+m_{1,1}u_1+m_{2,1}u_2)}{\dot{m}_{2,1}c_{2,1}}(\Delta T_p-\Delta T_{c1})$$
$$+\frac{2\pi r_{m,1}U_1m_{1,1}u_1}{\dot{m}_{2,1}c_{2,1}}(T_p-T_{c1})+\frac{2\pi r_{m,1}U_1m_{2,1}u_2}{\dot{m}_{2,1}c_{2,1}}(T_p-T_{c1})$$
$$+(1+u_2)\,\delta(x_1)\,\Delta T_6(t, L_6) \qquad (45)$$

with boundary condition $\Delta T_{c1}(t, 0)=0$.

Further simplifications give the following equation

$$2(1+u_2)\frac{\partial\Delta\bar{T}_{c1}}{\partial\iota}+a_2u_2\,e^{-(a_1+a_2)(\iota+1)/2}[\bar{T}_p(0,-1)-\bar{T}_{c1}(0,-1)]$$
$$+(1+f_{2,1})\frac{\partial\Delta\bar{T}_{c1}}{\partial\tau}$$
$$=a_2(1+m_{1,1}u_1+m_{2,1}u_2)(\Delta\bar{T}_p-\Delta\bar{T}_{c1})$$
$$+a_2m_{1,1}u_1\,e^{-(a_1+a_2)(\iota+1)/2}[\bar{T}_p(0,-1)-\bar{T}_{c1}(0,-1)]$$
$$+a_2m_{2,1}u_2\,e^{-(a_1+a_2)(\iota+1)/2}[\bar{T}_p(0,-1)-\bar{T}_{c1}(0,-1)]$$
$$+2(1+u_2)\,\delta(\iota+1)\,\Delta\bar{T}_6(\tau, 1) \qquad (46)$$

with boundary condition $\Delta\bar{T}_{c1}(\tau, -1)=0$.

Incorporating

$$\theta_6(\tau, 1)=\frac{\Delta \bar{T}_6(\tau, 1)}{\bar{T}_{c2}(0, -1)-\bar{T}_s(0, 1)}$$

and

$$\theta_2=\frac{\Delta \bar{T}_{c1}}{\bar{T}_p(0, -1)-\bar{T}_{c1}(0, -1)}$$

into Equation 46, Equation 10 is obtained.

The Equations 9 and 10 are solved by the MWR. The boundary conditions are shown in Equation 15a and the initial conditions in Equation 16. The residuals R_1 and R_2 are obtained by substituting the approximate solutions in Equation 19a for $i=1$ and 2 respectively.

$$\begin{aligned}R_1=&2(1+u_1)\frac{d}{d\iota}[(\iota+1)\mathrm{U}'(\iota)]\mathrm{X}_1+\bar{r}_1(1+f_{1,1})(\iota+1)\mathrm{U}'(\iota)\dot{\mathrm{X}}_1\\&-a_1(1+m_{1,1}u_1+m_{2,1}u_2)\mathrm{U}'(\iota)(\iota+1)(\mathrm{X}_2-\mathrm{X}_1)\\&+a_5(\iota+1)\mathrm{U}'(\iota)\mathrm{X}_1+a_1(m_{1,1}-1)\,e^{-(a_1+a_2)(\iota+1)/2}u_1\\&+a_1m_{2,1}\,e^{-(a_1+a_2)(\iota+1)/2}u_2-2(1+u_1)\,\delta(\iota+1)\theta_9(\tau, 1)\end{aligned} \quad (47)$$

$$\begin{aligned}R_2=&2(1+u_2)\frac{d}{d\iota}[(\iota+1)\mathrm{U}'(\iota)]\mathrm{X}_2+(1+f_{2,1})(\iota+1)\mathrm{U}'(\iota)\dot{\mathrm{X}}_2\\&-a_2(1+m_{1,1}u_1+m_{2,1}u_2)\mathrm{U}'(\iota)(\iota+1)(\mathrm{X}_1-\mathrm{X}_2)\\&-a_2(m_{2,1}-1)\,e^{-(a_1+a_2)(\iota+1)/2}u_2\\&-a_2m_{1,1}\,e^{-(a_1+a_2)(\iota+1)/2}u_1-2(1+u_2)\,\delta(\iota+1)\theta_6(\tau, 1)/\alpha\end{aligned} \quad (48)$$

The residuals R_1 and R_2 are orthogonalized with respect to $\mathrm{U}(\iota)$ to yield

$$\int_{-1}^{1}\mathrm{U}(\iota)R_1\,d\iota=0 \quad (49)$$

$$\int_{-1}^{1}\mathrm{U}(\iota)R_2\,d\iota=0 \quad (50)$$

Substituting Equations 47 and 48 to Equations 49 and 50, respectively, and evaluating integrals, Equations 25 and 26 are obtained.